Chapter 5: Exponents & Logs

Properties of Exponents

Exponent of Zero	$x^0 = 1$	$3^0 = 1$
Exponent of One	$x^1 = x$	$3^1 = 3$
Exponent Product	$x^a \cdot x^b = x^{a+b}$	$3^2 \cdot 3^5 = 3^7$
Exponent Quotient	$\frac{x^a}{x^b} = x^{a-b}$	$\frac{3^8}{3^2} = 3^6$
Exponent Power	$(x^a)^b = x^{ab}$	$(3^3)^4 = 3^{12}$
Power of a Product	$(ax)^b = a^b \cdot x^b$	$(3x)^4 = 3^4 \cdot x^4 = 81x^4$
Power of a Quotient	$\left(\frac{x}{a}\right)^b = \frac{x^b}{a^b}$	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2} = \frac{x^2}{9}$
Negative Exponent	$x^{-n} = \frac{1}{x^n}$	$x^{-4} = \frac{1}{x^4}$
Fractional Exponent	$x^{\frac{a}{b}} = \sqrt[b]{x^a}$	$x^{\frac{3}{4}} = \sqrt[4]{x^3}$

Simplifying Exponents without a Calculator

When you simplify an exponential expression without a calculator, do operations in this oWrder:

Negative sign: Move the value to the other side of the fraction

Root: The denominator of a fractional exponent is the root. Take the root of the number.

Power: The numerator of the fractional exponent is the power. Raise the number to the power.

Example: Simplify $125^{-2/3}$

Negative sign: $\frac{1}{125^{\frac{2}{3}}}$ Root: $\frac{1}{\left(\sqrt[3]{125}\right)^2} = \frac{1}{5^2}$ Power: $\frac{1}{25}$

Scientific Notation

Very large or very small numbers are often written in scientific notation. This is when it is changed to a decimal number times a power of ten. The decimal number is call the *mantissa*; the power of ten is called the *characteristic*.

Examples:

$4{,}380{,}000 = 4.38 \times 10^6$

$0.002347 = 2.347 \times 10^{-3}$

Using the EE function for Scientific Notation

When numbers are in scientific notation and you are using your calculator to do calculations, you should you the EE function ("Enter Exponent") to enter the numbers. The EE function "sticks" the two parts of the number (the mantissa and the characteristic) together, so your answer won't be affected by the order of operations.

Example:
To enter 4.38×10^6 in the calculator, type 4.38 then press the EE button. Now type 6.

Graphing Exponential Functions

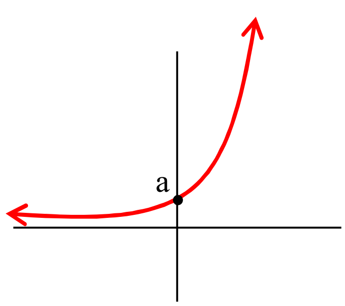
Exponential functions have the form: $$y = a(b)^x$$

a = y-intercept

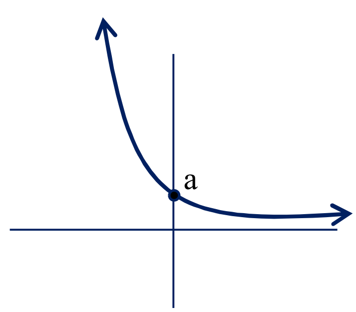
b = base

If the base is greater than 1, the function increases.

If the base is between 0 and 1, the function decreases.



b >1
Exponential growth



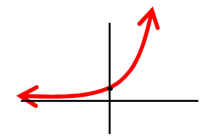
0 < b <1
Exponential decay

Writing Exponential Functions

Growth Functions

The formula for exponential growth can be written like this:

$$\text{Amount} = \text{Starting amount}\ (1 + \text{growth rate})^t$$



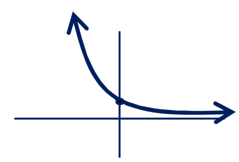
Amount = Total number at time t
Starting amount = The number of things you started with
Growth rate = Can be written as a fraction (1/4) or a decimal percent (12% = 0.12)
t = time

For example, if there are 40 ducks on a lake and the population of ducks increases by $\frac{1}{5}$ every year, the function that models this is: Amount of ducks = $40\ (1 + \frac{1}{5})^t$.

Decay Functions

The formula for exponential decay can be written like this:

$$\text{Amount} = \text{Starting amount}\ (1 - \text{decay rate})^t$$



Amount = Total number at time t
Starting amount = The number of things you started with
Decay rate = Can be written as a fraction (1/4) or a decimal percent (12% = 0.12)
t = time

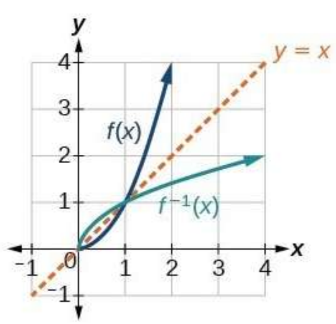
For example, if a bar of soap weighs 200 grams and its weight decreases by 4% every time it is used, the function that models this is: Weight of soap = $200\ (1 - 0.04)^t$.

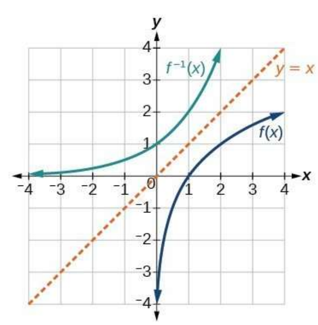
Inverse Functions

The inverse of a function "undoes" the function. The inverse of adding is subtracting; the inverse of multiplying is dividing; the inverse of squaring is a square root, etc. Not all functions have inverses.

If you are using f(x) notation, the inverse of a function is written as $f^{-1}(x)$, pronounced "the inverse of f". The superscript $^{-1}$ does NOT mean the reciprocal of f(x).

When a function and its inverse are plotted on the same graph, they form a reflection across the line y = x.





Finding the Inverse of a Function

To find the inverse of a function, exchange the *x* and *y*, then rearrange the equation to solve for *y*.

Example: Find the inverse of $y = x^2 - 3$.

$x = y^2 - 3$	Exchange the x and the y.
$x + 3 = y^2$	Solve for y.
$\sqrt{x+3} = y$	

The inverse of $f(x) = x^2 - 3$ is equal to $f^{-1}(x) = \sqrt{x+3}$.

Logarithms

A logarithm is the inverse of an exponent. You will use them to solve equations.

Logarithms are written like this: $\log_b y = x$

b and *y* must be positive numbers

This equation is pronounced "log base b of y equals x".

The LOG button on your calculator uses a base of 10, since our number system is decimal (base 10).

Solving Logs without a Calculator

To solve logs without a calculator, we will "scoot" the base, and change the problem to one with exponents.

$$\log_b y = x \quad \text{becomes } y = b^x$$

"scoot"

Example:

$\log_5 x = 2$	
$x = 5^2$	"Scoot" the base 5 to the other side.
$x = 25$	Simplify the expression.

Properties of Logarithms

Product	$\log_b(xy) = \log_b x + \log_b y$	$\log_3(4x) = \log_3 4 + \log_3 x$
Quotient	$\log_b \frac{x}{y} = \log_b x - \log_b y$	$\log_7 \frac{x}{2} = \log_7 x - \log_7 2$
Power	$\log_b x^n = n \log_b x$	$\log_4 x^5 = 5 \log_4 x$
Change of Base	$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_6 75 = \frac{\log 75}{\log 6}$

A Special Base

Logarithms with the "log" notation are called *common logarithms*.
A *natural logarithm* is a special logarithm because it always has a base of *e*.
e is called Euler's number. It's approximately equal to 2.718282…

Euler's number is used in all kinds of functions for natural or continuous growth or decay, like when you're trying to calculate the number of bacteria on an agar plate or the amount of radioactive mass left after a certain amount of time.

Natural logs follow the same rules as common logs. We use the notation $\ln x$ to represent $\log_e x$.
There is an *ln x* button on your calculator.

Solving Exponential Equations

When the variable you are solving for is an exponent, you have to use logs to solve for it.

Example: $45 = 3(1.2)^x$

$15 = 1.2^x$	Divide both sides by 3 to get the exponential term alone.
$\log 15 = \log 1.2^x$	Take a log of both sides.
$\log 15 = x \log 1.2$	Use the Power Rule
$\frac{\log 15}{\log 1.2} = x$	Divide to solve for x.
$14.852 \approx x$	Use the calculator to get a value.

Example: $7 = 2e^x$

$3.5 = e^x$	Divide both sides by 2 to get the exponential term alone.
$\ln 3.5 = \ln e^x$	Take a natural log of both sides. Always use a natural log when the base is e.
$\ln 3.5 = x \ln e$	Use the Power Rule
$\ln 3.5 = x$	The natural log of e equals 1. $\ln e = 1$
$1.252 \approx x$	Use the calculator to get a value.

Solving Logarithmic Equations

Logs with Same Base on Both Sides.

If an equation has two logs of the same base or two ln that are equal, that means their arguments are equal. Just eliminate the logs and solve for x.

Example: $\log_4(3x - 7) = \log_4(x + 9)$

$$3x - 7 = x + 9$$
$$2x - 7 = 9$$
$$2x = 16$$
$$x = 8$$

The solution is 8. Always check your solution to make sure the argument of each log is not negative when you plug the solution in.

Log equal to a Number

When the equation has a log or ln equal to a number, just rearrange the equation to make it an exponential equation.

Example: $\log_2(3x - 1) = 3$

$\log_2(3x - 1) = 3$ "Scoot" the base.

$$3x - 1 = 2^3$$
$$3x - 1 = 8$$
$$3x = 9$$
$$x = 3$$

Real-World Applications

Compound interest
Bacterial growth
Radioactive decay
Etc.

PRACTICE

Simplify each expression so that all exponents are positive. Do not use a calculator.

1. -4^2
2. $(-4)^2$
3. $4^0 \cdot 2^{-3}$
4. $4^{-2} \cdot 4^3$
5. $\frac{(2^3)^3}{(2^2)^3}$
6. $\left(\frac{9}{2}\right)^{-2}$
7. $\frac{3^{-2}}{2}$
8. $\frac{2^3 \cdot 3^2}{2^4 \cdot 3^{-2}}$
9. $(3x^5)^4$
10. $(12x)(3x)^2$
11. $\frac{(3x^4)^3}{18x^5}$
12. $\left(\frac{x^4}{y^5}\right)^3 \left(\frac{y^6}{x^5}\right)^4$
13. $\left(\frac{8x^{11}y^7}{4x^3y^5}\right)^4$

Evaluate without using a calculator.

14. $\frac{5^{400}}{5^{397}}$
15. $\frac{9 \cdot 4^{2026}}{2 \cdot 4^{2024}}$
16. $\frac{(3^{1081})^{10}}{(3^{1201})^9}$

Simply each expression without using a calculator.

17. $8^{2/3}$
18. $(-27)^{2/3}$
19. $(-8)^{-5/3}$
20. $25^{-3/2}$
21. $\left(\frac{27}{8}\right)^{2/3}$

Use a calculator to evaluate each expression. Write your answer in scientific notation.

22. $1.2 \times 10^8 / 3 \times 10^5$
23. $6 \times 10^{-4} / 1.5 \times 10^3$
24. $2.5 \times 10^{16} / 5 \times 10^{-11}$

For each function: a. List the y-intercept
b. Indicate whether it is increasing or decreasing

25. $y = 42(1.25)^x$
26. $f(x) = 0.8(20)^x$
27. $g(x) = 450(\frac{3}{4})^x$
28. $y = \frac{1}{2}(0.489)^x$

Write an exponential function for each problem. $y = \text{Starting amount } (1 + \text{growth rate})^t$
or $y = \text{Starting amount } (1 - \text{decay rate})^t$

29. The height of a plant is 8 inches, and it increase by 5% every week.
30. You deposit $250 in the bank, and it earns 3% interest every year.
31. 2000 bacteria were living on a Petri dish, and their population increased by 18% every day.
32. A beach contains 42 tons of sand, and the amount of sand decreased by 2% every time a wave goes out.
33. A runner's time to finish a race was 83 seconds, and it decreased by 12% every week he practiced.
34. A chunk of uranium had a mass of 24 grams, and its mass decreased by ¼ every year.

For each problem: a. Write an exponential function to model the problem.
b. Solve for the y-values it asks for.

35. You invest $5000 in a fund that earns 7% interest. How much money is in your account 8 years later?
36. A radioactive rock loses 4% of its mass every hour. The rock has a mass of 80 grams. What is its mass 12 hours later?
37. There are 120 bacteria on a Petri dish. Their population grows 36% every day. How many bacteria are there after four days?
38. There are 5300 rural acres in Comal County. The acreage decreases by $\frac{3}{100}$ each year. How many rural acres will there be in 15 years?
39. Your little brother is 32 inches tall. If his height increases by $\frac{1}{12}$ every year. How tall will he be in 10 years?

Find the equation for the inverse of each function.

40. $f(x) = 2x + 7$
41. $f(x) = (x - 5)^2$
42. $f(x) = \sqrt[3]{x} + 2$
43. $f(x) = \log_2 x$

Find the argument x of the logarithm.

44. $\log_5 x = 2$

Find the logarithm x.

45. $\log_2 8 = x$

Find the base x of the logarithm.

46. $\log_x 16 = 4$

Use a calculator to evaluate the logarithm.

47. $\log 24$

48. $\log 5.3$

49. $\log 0.256$

50. $\log 10$

51. $\log 100$

52. $\log 100000$

53. $\log_2 9$

54. $\log_6 154$

55. $\log_5 500$

56. $\ln 8$

57. $\ln 40.6$

58. $\ln e$