Section 11-2 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

- The Derivative of e^x
- The Derivative of ln x
- Other Logarithmic and Exponential Functions
- Exponential and Logarithmic Models

In this section, we obtain formulas for the derivatives of logarithmic and exponential functions. A review of Sections 2-4 and 2-5 may prove helpful. In particular, recall that $f(x) = e^x$ is the exponential function with base $e \approx 2.718$ and the inverse of the function e^x is the natural logarithm function ln x. More generally, if b is a positive real number, $b \neq 1$, then the exponential function b^x with base b, and the logarithmic function $\log_b x$ with base b, are inverses of each other.

The Derivative of e^x

In the process of finding the derivative of e^x, we will use (without proof) the fact that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1 \tag{1}$$

Explore & Discuss 1 Complete Table 1.

TABLE 1

h	−0.1	−0.01	−0.001 → 0 ← 0.001	0.01	0.1
$\frac{e^h - 1}{h}$					

Do your calculations make it reasonable to conclude that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1?$$

Discuss.

We now apply the four-step process (Section 10-4) to the exponential function $f(x) = e^x$.

Step 1. Find $f(x + h)$.

$$f(x + h) = e^{x+h} = e^x e^h \quad \text{See Section 2-4.}$$

Step 2. Find $f(x + h) - f(x)$.

$$\begin{aligned} f(x + h) - f(x) &= e^x e^h - e^x \quad \text{Factor out } e^x. \\ &= e^x(e^h - 1) \end{aligned}$$

Step 3. Find $\frac{f(x + h) - f(x)}{h}$.

$$\frac{f(x + h) - f(x)}{h} = \frac{e^x(e^h - 1)}{h} = e^x\left(\frac{e^h - 1}{h}\right)$$

Step 4. Find $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right)$$
$$= e^x \lim_{h\to 0}\left(\frac{e^h-1}{h}\right) \qquad \text{Use the limit in (1).}$$
$$= e^x \cdot 1 = e^x$$

Thus,

$$\frac{d}{dx}e^x = e^x \qquad \text{The derivative of the exponential function is the exponential function.}$$

EXAMPLE 1 **Finding Derivatives** Find $f'(x)$ for

(A) $f(x) = 5e^x - 3x^4 + 9x + 16$
(B) $f(x) = -7x^e + 2e^x + e^2$

SOLUTIONS (A) $f'(x) = 5e^x - 12x^3 + 9$
(B) $f'(x) = -7ex^{e-1} + 2e^x$

Remember that e is a real number, so the power rule (Section 10–5) is used to find the derivative of x^e. The derivative of the exponential function e^x, however, is e^x. Note that $e^2 \approx 7.389$ is a constant, so its derivative is 0.

MATCHED PROBLEM 1 Find $f'(x)$ for

(A) $f(x) = 4e^x + 8x^2 + 7x - 14$
(B) $f(x) = x^7 - x^5 + e^3 - x + e^x$

CAUTION:

$$\frac{d}{dx}e^x \neq xe^{x-1} \qquad \frac{d}{dx}e^x = e^x$$

The power rule cannot be used to differentiate the exponential function. The power rule applies to exponential forms x^n where the exponent is a constant and the base is a variable. In the exponential form e^x, the base is a constant and the exponent is a variable.

The Derivative of ln *x*

We summarize some important facts about logarithmic functions from Section 2–5:

SUMMARY Recall that the inverse of an exponential function is called a **logarithmic function.** For $b > 0$ and $b \neq 1$,

Logarithmic form		Exponential form
$y = \log_b x$	is equivalent to	$x = b^y$
Domain: $(0, \infty)$		Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$		Range: $(0, \infty)$

The graphs of $y = \log_b x$ and $y = b^x$ are symmetric with respect to the line $y = x$. (See Figure 1.)

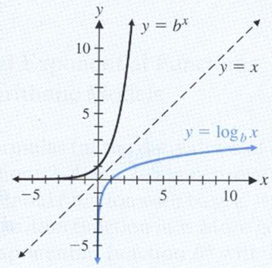

FIGURE 1

Of all the possible bases for logarithmic functions, the two most widely used are

$$\log x = \log_{10} x \quad \text{Common logarithm (base 10)}$$
$$\ln x = \log_e x \quad \text{Natural logarithm (base } e\text{)}$$

We are now ready to use the definition of the derivative and the four-step process discussed in Section 10-4 to find a formula for the derivative of $\ln x$. Later we will extend this formula to include $\log_b x$ for any base b.

Let $f(x) = \ln x, x > 0$.

Step 1. Find $f(x + h)$.

$$f(x + h) = \ln(x + h) \quad \ln(x+h) \text{ cannot be simplified.}$$

Step 2. Find $f(x + h) - f(x)$.

$$f(x + h) - f(x) = \ln(x + h) - \ln x \quad \text{Use } \ln A - \ln B = \ln\frac{A}{B}.$$
$$= \ln\frac{x + h}{x}$$

Step 3. Find $\frac{f(x + h) - f(x)}{h}$.

$$\frac{f(x + h) - f(x)}{h} = \frac{\ln(x + h) - \ln x}{h}$$
$$= \frac{1}{h}\ln\frac{x + h}{x} \quad \text{Multiply by } 1 = x/x \text{ to change form.}$$
$$= \frac{x}{x}\cdot\frac{1}{h}\ln\frac{x + h}{x}$$
$$= \frac{1}{x}\left[\frac{x}{h}\ln\left(1 + \frac{h}{x}\right)\right] \quad \text{Use } p \ln A = \ln A^p.$$
$$= \frac{1}{x}\ln\left(1 + \frac{h}{x}\right)^{x/h}$$

Step 4. Find $f'(x) = \lim_{h\to 0}\frac{f(x + h) - f(x)}{h}$.

$$f'(x) = \lim_{h\to 0}\frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h\to 0}\left[\frac{1}{x}\ln\left(1 + \frac{h}{x}\right)^{x/h}\right] \quad \text{Let } s = h/x. \text{ Note that } h \to 0 \text{ implies } s \to 0.$$

$$= \frac{1}{x}\lim_{s\to 0}[\ln(1+s)^{1/s}] \quad \text{Use a new limit property.*}$$

$$= \frac{1}{x}\ln\left[\lim_{s\to 0}(1+s)^{1/s}\right] \quad \text{Use the definition of } e.$$

$$= \frac{1}{x}\ln e \quad \ln e = \log_e e = 1$$

$$= \frac{1}{x}$$

Thus,

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

INSIGHT

In the derivation of the derivative of $\ln x$, we used the following properties of logarithms:

$$\ln\frac{A}{B} = \ln A - \ln B \qquad \ln A^p = p \ln A$$

We also noted that there is no property that simplifies $\ln(A + B)$. (See Theorem 1 in Section 2-5 for a list of properties of logarithms.)

EXAMPLE 2 **Finding Derivatives** Find y' for

(A) $y = 3e^x + 5 \ln x$
(B) $y = x^4 - \ln x^4$

SOLUTIONS (A) $y' = 3e^x + \frac{5}{x}$

(B) Before taking the derivative, we use a property of logarithms (see Theorem 1, Section 2-5) to rewrite y.

$$y = x^4 - \ln x^4 \quad \text{Use } \ln M^p = p \ln M.$$

$$y = x^4 - 4\ln x \quad \text{Now take the derivative of both sides.}$$

$$y' = 4x^3 - \frac{4}{x}$$

MATCHED PROBLEM 2 Find y' for

(A) $y = 10x^3 - 100 \ln x$
(B) $y = \ln x^5 + e^x - \ln e^2$

Other Logarithmic and Exponential Functions

In most applications involving logarithmic or exponential functions, the number e is the preferred base. However, in some situations it is convenient to use a base other than e. Derivatives of $y = \log_b x$ and $y = b^x$ can be obtained by expressing these functions in terms of the natural logarithmic and exponential functions.

* The following new limit property is used: If $\lim_{x\to c} f(x)$ exists and is positive, then $\lim_{x\to c}[\ln f(x)] = \ln[\lim_{x\to c} f(x)]$.

We begin by finding a relationship between $\log_b x$ and $\ln x$ for any base b, $b > 0$ and $b \neq 1$. Some of you may prefer to remember the process, others the formula.

$$y = \log_b x \qquad \text{Change to exponential form.}$$
$$b^y = x \qquad \text{Take the natural logarithm of both sides.}$$
$$\ln b^y = \ln x \qquad \text{Recall that } \ln b^y = y \ln b.$$
$$y \ln b = \ln x \qquad \text{Solve for } y.$$
$$y = \frac{1}{\ln b} \ln x$$

Thus,

$$\log_b x = \frac{1}{\ln b} \ln x \qquad \text{Change-of-base formula for logarithms*} \qquad (2)$$

Similarly, we can find a relationship between b^x and e^x for any base b, $b > 0$, $b \neq 1$.

$$y = b^x \qquad \text{Take the natural logarithm of both sides.}$$
$$\ln y = \ln b^x \qquad \text{Recall that } \ln b^x = x \ln b.$$
$$\ln y = x \ln b \qquad \text{Take the exponential function of both sides.}$$
$$y = e^{x \ln b}$$

Thus,

$$b^x = e^{x \ln b} \qquad \text{Change-of-base formula for exponential functions} \qquad (3)$$

Differentiating both sides of equation (2) gives

$$\frac{d}{dx} \log_b x = \frac{1}{\ln b} \frac{d}{dx} \ln x = \frac{1}{\ln b}\left(\frac{1}{x}\right)$$

It can be shown that the derivative of the function e^{cx}, where c is a constant, is the function ce^{cx} (see Problems 49–50 in Exercise 11-2 or the more general results of Section 11-4). Therefore, differentiating both sides of equation (3), we have

$$\frac{d}{dx} b^x = e^{x \ln b} \ln b = b^x \ln b$$

For convenient reference, we list the derivative formulas that we have obtained for exponential and logarithmic functions:

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

For $b > 0$, $b \neq 1$,

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} b^x = b^x \ln b$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad \frac{d}{dx} \log_b x = \frac{1}{\ln b}\left(\frac{1}{x}\right)$$

EXAMPLE 3 **Finding Derivatives** Find $g'(x)$ for

(A) $g(x) = 2^x - 3^x$

(B) $g(x) = \log_4 x^5$

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* Equation (2) is a special case of the **general change-of-base formula** for logarithms (which can be derived in the same way): $\log_b x = (\log_a x)/(\log_a b)$.

SOLUTIONS (A) $g'(x) = 2^x \ln 2 - 3^x \ln 3$

(B) First use a property of logarithms to rewrite $g(x)$.

$$g(x) = \log_4 x^5 \quad \text{Use } \log_b M^p = p \log_b M.$$
$$g(x) = 5 \log_4 x \quad \text{Take the derivative of both sides.}$$
$$g'(x) = \frac{5}{\ln 4}\left(\frac{1}{x}\right)$$

MATCHED PROBLEM 3 Find $g'(x)$ for

(A) $g(x) = x^{10} + 10^x$

(B) $g(x) = \log_2 x - 6 \log_5 x$

Explore & Discuss **2**

(A) The graphs of $f(x) = \log_2 x$ and $g(x) = \log_4 x$ are shown in Figure 2. Which graph belongs to which function?

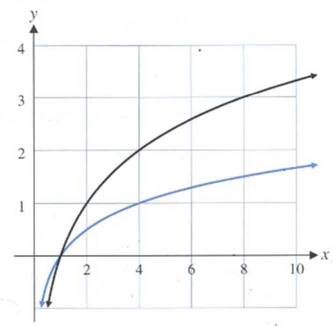

FIGURE 2

(B) Sketch graphs of $f'(x)$ and $g'(x)$.

(C) The function $f(x)$ is related to $g(x)$ in the same way that $f'(x)$ is related to $g'(x)$. What is that relationship?

Exponential and Logarithmic Models

EXAMPLE 4 **Price–Demand Model** An Internet store sells blankets made of the finest Australian wool. If the store sells x blankets at a price of $\$p$ per blanket, then the price–demand equation is $p = 350(0.999)^x$. Find the rate of change of price with respect to demand when the demand is 800 blankets, and interpret the result.

SOLUTION $\frac{dp}{dx} = 350(0.999)^x \ln 0.999$

If $x = 800$, then

$$\frac{dp}{dx} = 350(0.999)^{800} \ln 0.999 \approx -0.157, \text{ or } -\$0.16$$

When the demand is 800 blankets, the price is decreasing about \$0.16 per blanket.

MATCHED PROBLEM 4 The store in Example 4 also sells a reversible fleece blanket. If the price–demand equation for reversible fleece blankets is $p = 200(0.998)^x$, find the rate of change of price with respect to demand when the demand is 400 blankets and interpret the result.

EXAMPLE 5 **Cable TV Subscribers** A statistician used data from the U.S. census to construct the model

$$S(t) = 21 \ln t + 2$$

where $S(t)$ is the number of cable TV subscribers (in millions) in year t ($t = 0$ corresponds to 1980). Use this model to estimate the number of cable TV subscribers in 2010 and the rate of change of the number of subscribers in 2010 (round both to the nearest tenth of a million). Interpret these results.

SOLUTION Since 2010 corresponds to $t = 30$, we must find $S(30)$ and $S'(30)$.

$$S(30) = 21 \ln 30 + 2 = 73.4 \text{ million}$$

$$S'(t) = 21\frac{1}{t} = \frac{21}{t}$$

$$S'(30) = \frac{21}{30} = 0.7 \text{ million}$$

In 2010 there will be approximately 73.4 million subscribers, and this number is growing at the rate of 0.7 million per year.

MATCHED PROBLEM 5 A model for newspaper circulation is

$$C(t) = 83 - 9 \ln t$$

where $C(t)$ is newspaper circulation (in millions) in year t ($t = 0$ corresponds to 1980). Use this model to estimate the circulation and the rate of change of circulation in 2010 (round both to the nearest tenth of a million). Interpret these results.

INSIGHT

On most graphing calculators, exponential regression produces a function of the form $y = a \cdot b^x$. Formula (3) enables you to change the base b (chosen by the graphing calculator) to the more familiar base e:

$$y = a \cdot b^x = a \cdot e^{x \ln b}$$

On most graphing calculators, logarithmic regression produces a function of the form $y = a + b \ln x$. Formula (2) enables you to write the function in terms of logarithms to any base d that you may prefer:

$$y = a + b \ln x = a + b(\ln d) \log_d x$$

Answers to Matched Problems

1. (A) $4e^x + 16x + 7$
   (B) $7x^6 - 5x^4 - 1 + e^x$
2. (A) $30x^2 - \frac{100}{x}$
   (B) $\frac{5}{x} + e^x$
3. (A) $10x^9 + 10^x \ln 10$
   (B) $\left(\frac{1}{\ln 2} - \frac{6}{\ln 5}\right)\frac{1}{x}$
4. The price is decreasing at the rate of \$0.18 per blanket.
5. The circulation in 2010 is approximately 52.4 million and is decreasing at the rate of 0.3 million per year.

Exercise 11-2

A *In Problems 1–14, find $f'(x)$.*

1. $f(x) = 5e^x + 3x + 1$

2. $f(x) = -7e^x - 2x + 5$

3. $f(x) = -2 \ln x + x^2 - 4$

4. $f(x) = 6 \ln x - x^3 + 2$

5. $f(x) = x^3 - 6e^x$

6. $f(x) = 9e^x + 2x^2$

7. $f(x) = e^x + x - \ln x$

8. $f(x) = \ln x + 2e^x - 3x^2$

9. $f(x) = \ln x^3$

10. $f(x) = \ln x^8$

11. $f(x) = 5x - \ln x^5$

12. $f(x) = 4 + \ln x^9$

13. $f(x) = \ln x^2 + 4e^x$

14. $f(x) = \ln x^{10} + 2 \ln x$

B *In Problems 15–22, find the equation of the line tangent to the graph of f at the indicated value of x.*

15. $f(x) = 3 + \ln x;\ x = 1$

16. $f(x) = 2 \ln x;\ x = 1$

17. $f(x) = 3e^x;\ x = 0$

18. $f(x) = e^x + 1;\ x = 0$

19. $f(x) = \ln x^3;\ x = e$

20. $f(x) = 1 + \ln x^4;\ x = e$

21. $f(x) = 2 + e^x;\ x = 1$

22. $f(x) = 5e^x;\ x = 1$

23. A student claims that the line tangent to the graph of $f(x) = e^x$ at $x = 3$ passes through the point (2, 0) (see the figure). Is she correct? Will the line tangent at $x = 4$ pass through (3, 0)? Explain.

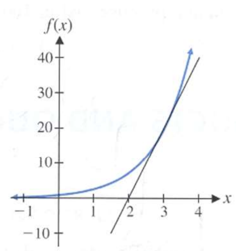

Figure for 23

24. Refer to Problem 23. Does the line tangent to the graph of $f(x) = e^x$ at $x = 1$ pass through the origin? Are there any other lines tangent to the graph of f that pass through the origin? Explain.

25. A student claims that the line tangent to the graph of $g(x) = \ln x$ at $x = 3$ passes through the origin (see the figure). Is he correct? Will the line tangent at $x = 4$ pass through the origin? Explain.

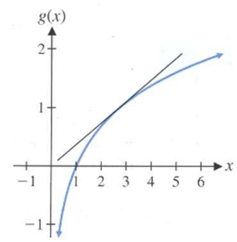

Figure for 25

26. Refer to Problem 25. Does the line tangent to the graph of $f(x) = \ln x$ at $x = e$ pass through the origin? Are there any other lines tangent to the graph of f that pass through the origin? Explain.

In Problems 27–30, first use appropriate properties of logarithms to rewrite f(x), and then find $f'(x)$.

27. $f(x) = 10x + \ln 10x$

28. $f(x) = 2 + 3 \ln \dfrac{1}{x}$

29. $f(x) = \ln \dfrac{4}{x^3}$

30. $f(x) = x + 5 \ln 6x$

C *In Problems 31–42, find $\dfrac{dy}{dx}$ for the indicated function y.*

31. $y = \log_2 x$

32. $y = 3 \log_5 x$

33. $y = 3^x$

34. $y = 4^x$

35. $y = 2x - \log x$

36. $y = \log x + 4x^2 + 1$

37. $y = 10 + x + 10^x$

38. $y = x^5 - 5^x$

39. $y = 3 \ln x + 2 \log_3 x$

40. $y = -\log_2 x + 10 \ln x$

41. $y = 2^x + e^2$

42. $y = e^3 - 3^x$

In Problems 43–48, use graphical approximation methods to find the points of intersection of f(x) and g(x) (to two decimal places).

43. $f(x) = e^x;\ g(x) = x^4$
[Note that there are three points of intersection and that e^x is greater than x^4 for large values of x.]

44. $f(x) = e^x$; $g(x) = x^5$

[Note that there are two points of intersection and that e^x is greater than x^5 for large values of x.]

45. $f(x) = (\ln x)^2$; $g(x) = x$

46. $f(x) = (\ln x)^3$; $g(x) = \sqrt{x}$

47. $f(x) = \ln x$; $g(x) = x^{1/5}$

48. $f(x) = \ln x$; $g(x) = x^{1/4}$

49. Explain why $\lim_{h \to 0} \frac{e^{ch} - 1}{h} = c$.

50. Use the result of Problem 49 and the four-step process to show that if $f(x) = e^{cx}$, then $f'(x) = ce^{cx}$.

Applications

51. *Salvage value.* The salvage value S (in dollars) of a company airplane after t years is estimated to be given by

$$S(t) = 300{,}000(0.9)^t$$

What is the rate of depreciation (in dollars per year) after 1 year? 5 years? 10 years?

52. *Resale value.* The resale value R (in dollars) of a company car after t years is estimated to be given by

$$R(t) = 20{,}000(0.86)^t$$

What is the rate of depreciation (in dollars per year) after 1 year? 2 years? 3 years?

53. *Bacterial growth.* A single cholera bacterium divides every 0.5 hour to produce two complete cholera bacteria. If we start with a colony of 5,000 bacteria, after t hours there will be

$$A(t) = 5{,}000 \cdot 2^{2t} = 5{,}000 \cdot 4^t$$

bacteria. Find $A'(t)$, $A'(1)$, and $A'(5)$, and interpret the results.

54. *Bacterial growth.* Repeat Problem 53 for a starting colony of 1,000 bacteria such that a single bacterium divides every 0.25 hour.

55. *Blood pressure.* An experiment was set up to find a relationship between weight and systolic blood pressure in normal children. Using hospital records for 5,000 normal children, the experimenters found that the systolic blood pressure was given approximately by

$$P(x) = 17.5(1 + \ln x) \qquad 10 \le x \le 100$$

where $P(x)$ is measured in millimeters of mercury and x is measured in pounds. What is the rate of change of blood pressure with respect to weight at the 40-pound weight level? At the 90-pound weight level?

56. Refer to Problem 55. Find the weight (to the nearest pound) at which the rate of change of blood pressure with respect to weight is 0.3 millimeter of mercury per pound.

57. *Psychology: stimulus/response.* In psychology, the Weber–Fechner law for the response to a stimulus is

$$R = k \ln \frac{S}{S_0}$$

where R is the response, S is the stimulus, and S_0 is the lowest level of stimulus that can be detected. Find dR/dS.

58. *Psychology: learning.* A mathematical model for the average of a group of people learning to type is given by

$$N(t) = 10 + 6 \ln t \qquad t \ge 1$$

where $N(t)$ is the number of words per minute typed after t hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?

Section 11-3 DERIVATIVES OF PRODUCTS AND QUOTIENTS

- Derivatives of Products
- Derivatives of Quotients

The derivative properties discussed in Section 10-5 added substantially to our ability to compute and apply derivatives to many practical problems. In this and the next two sections, we add a few more properties that will increase this ability even further.

Derivatives of Products

In Section 10-5, we found that the derivative of a sum is the sum of the derivatives. Is the derivative of a product the product of the derivatives?